MODULE - II

1 2

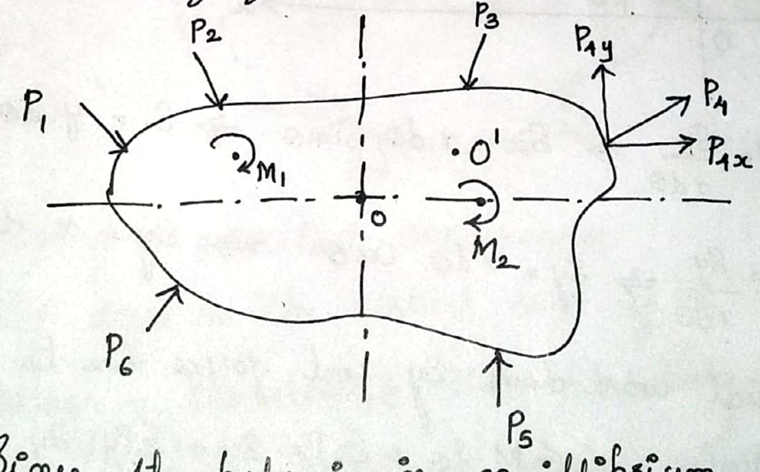
Virtual work:-

Virtual means the effect exists but actually, it is not the fact

The virtual work may be defined as the workdone by real forces due to hypothetical displacements or the work done by the hypothetical forces during real displacements

Bernaulli's Principle of Virtual displacement

Consider a rigid body subjected to P system of forces and M system of moments. Let P_x, & P_y be the compoments of force in x direction & y direction.



Since the body is in equillibrium,

$$\Sigma P_x = 0$$

$$\Sigma P_y = 0$$

$$\Sigma M + \Sigma P_x y + \Sigma P_y . x = 0$$

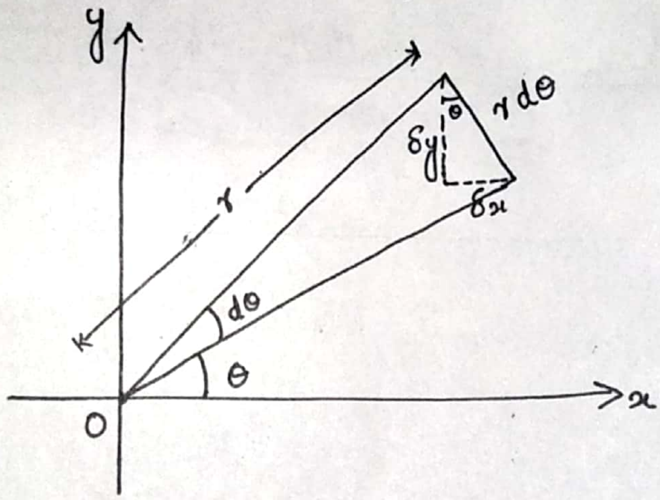
Suppose a virtual displacement OO' is given to the rigid body. Let the compoment of OO' in x-direction be δx and in y- direction be δy.

$\therefore$ Virtual workdone $= \Sigma P_x . \delta x + \Sigma P_y . \delta y$ $\quad \because \Sigma P_x = 0, \ \Sigma P_y = 0$

$= \delta x \, \Sigma P_x + \delta y \, \Sigma P_y$

Virtual work done $= \underline{\underline{0}}$

Now consider the rotation of the rigid body by virtual ~~dis~~ rotation '$d\theta$'. If a point $Q(x,y)$ is at distance r from the origin,



$\sin\theta = y/r$

$\cos\theta = x/r$

$\sin\theta = \frac{\delta x}{r d\theta} \Rightarrow \delta x = r \, d\theta \, \sin\theta \Rightarrow \delta = y . d\theta$

$\cos\theta = \frac{\delta y}{r d\theta} \Rightarrow \delta y = r \, d\theta \, \cos\theta \Rightarrow \delta y = x . d\theta$

$\therefore$ The virtual work done by real forces due to virtual displacements $= \Sigma M \, d\theta + \Sigma P_x . \delta x + \Sigma P_y . \delta y$

$= \Sigma M \, d\theta + \Sigma P_x . y d\theta + \Sigma P_y . x d\theta$

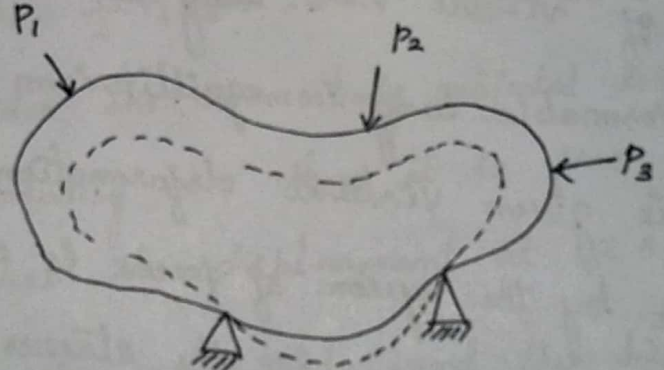
$= d\theta \left[\Sigma M + \Sigma P_x . y + \Sigma P_y . x \right]$

Virtual work $= \underline{\underline{0}}$

Bernaulli's Principle states that "If a body is in equillibrium under the system of forces and/or moments, the virtual workdone by this system of forces and/or moments during virtual displacement is zero".

Principle of virtual work for Deformable Bodies.

Consider an elastic body subjected to a system of forces. These forces causes real internal stresses and every element in the body is in equillibrium under the action of external forces and internal stresses.



Imagine a virtual displacement.

Let dW_0 be the virtual work of the external forces acting on the element. Since the body is elastic, there will be virtual displacement and also virtual deformation. Hence the external work applied to the element is dissipated in two forms.

1. The virtual work, dW_r, treating the element as rigid body
2. The virtual work of deformation of the element dW_i

This may be called as virtual strain energy of the element

$$dW_\theta = dW_i + dW_r$$

From the principle of virtual displacement

$$dW_r = 0$$

$$\Rightarrow dW_\theta = dW_i$$

Integrate over the body

$$W_\theta = W_i$$

where W_θ = Total virtual work done by the force system

W_i = Internal virtual strain energy of the entire body

Principle of virtual work may be states that

"If a deformable body in equilibrium under a system of forces is given virtual deformation the virtual work done by the system of forces is equal to the internal virtual work done by the stresses due to that system of forces".

<u>Unit Load Method:-</u>

Consider a body which is subjected to forces $P_1, P_2, P_3, \ldots P_n$ applied gradually. Let displacement under load points be $\Delta_1, \Delta_2, \Delta_3, \ldots \Delta_n$ and at point c be Δ. Then

External work done $= \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \ldots + \frac{1}{2}P_n\Delta_n$

strain energy stored $= \int \frac{1}{2} p e \, dv$

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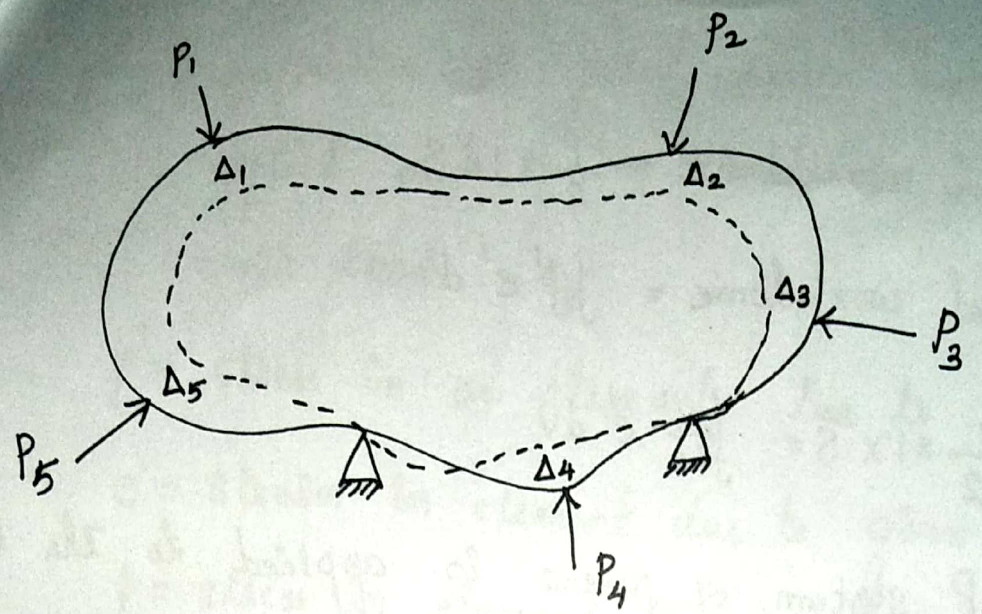


A body subjected to load and deformed shape.

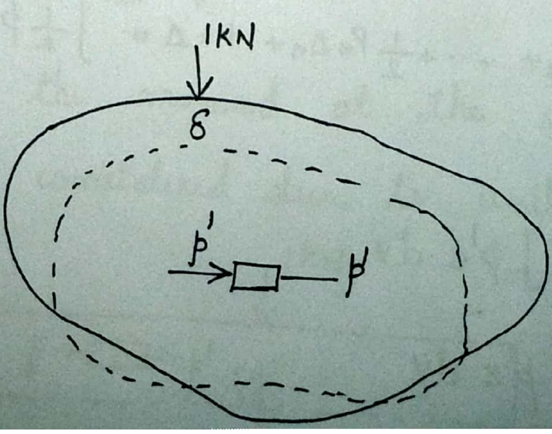
where p is stress & e is the strain in the element considered.

$$EW = SE$$

$$\Rightarrow \frac{1}{2} P_1 \Delta_1 + \frac{1}{2} P_2 \Delta_2 + \cdots\cdots + \frac{1}{2} P_n \Delta_n = \int \frac{1}{2} p\, e\, dv \quad \text{—} (1)$$

Now, consider the same body subjected to an unit load applied gradually at c when it is free of system of P forces. Let the displacement at 1, 2, 3 .... n be $\delta_1, \delta_2, \delta_3, \ldots \delta_n$ respectively and the displacement at c be δ.

Let the stress produced in the element be p' and the strain be e'. Then



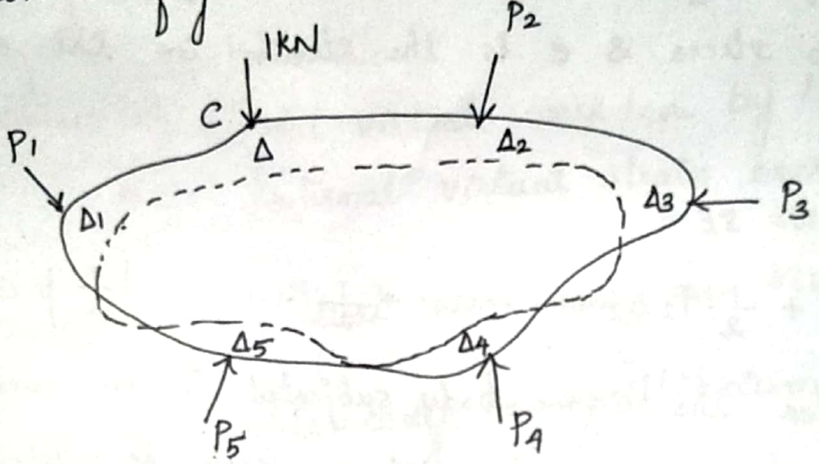
Then

External workdone $= \frac{1}{2} \times 1 \times \delta$

Internal workdone $= \int \frac{1}{2} p' e' \, dv$

$\therefore \frac{1}{2} \times 1 \times \delta = \int \frac{1}{2} p' . e' \, dv$

Now if P system of forces is applied to the body shown in fig.



External workdone $= \frac{1}{2} P_1 \Delta_1 + \frac{1}{2} P_2 \Delta_2 + \dots + \frac{1}{2} P_n \Delta_n + 1 \times \Delta$.

$(1 \times \Delta)$ since unit load is already acting.

Internal workdone $= \int \frac{1}{2} p e \, dv + \int p' e \, dv$.

Since the stress p' is acting throughout the deformat^n

$\therefore$ EW = IW

$$\frac{1}{2} P_1 \Delta_1 + \frac{1}{2} P_2 \Delta_2 + \dots + \frac{1}{2} P_n \Delta_n + 1 \times \Delta = \int \frac{1}{2} p e \, dv + \int p' e \, dv \quad \text{—} (2)$$

eqn (2) − (1)

$$1 \times \Delta = \int p' e \, dv$$

$$\Delta = \int p' e \, dv$$

where Δ = deflection at point where unit load is applied and is measured in the direction of unit load.

p' = stress in an element due to unit load.

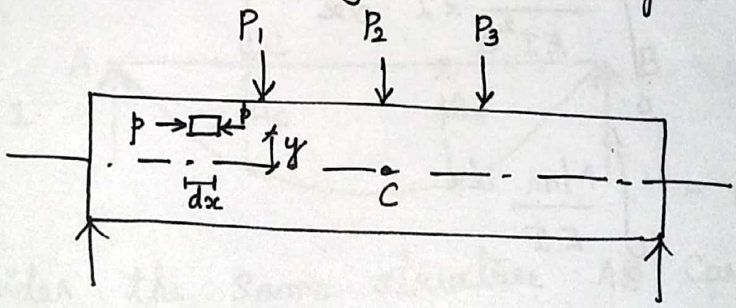
e = strain in element due to given load system

p = stress in element due to given load system

This eqn is the basis for the unit load method

Unit Load method:-

Consider a beam subjected to a system of P forces.

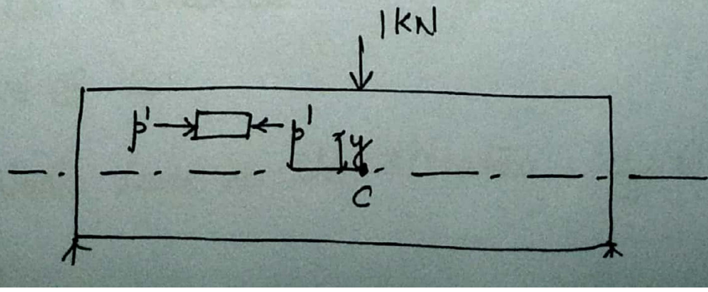


The stress in the element at a distance y from NA

$$p = \frac{M}{I} \cdot y \quad \text{where } M = \text{moment}.$$

Strain, $e = \dfrac{M}{EI} y$.

Let m be the moment at the section where the element is considered, due to unit load acting at C



stress $p' = \frac{my}{I}$

$\therefore$ we know that, $\Delta = \int p' e \, dv$

$$\Delta = \int \frac{my}{I} \times \frac{M}{EI} y \, dv$$

$$= \int \frac{Mm}{EI^2} y^2 \, dv$$

$$= \int_0^L \frac{Mm}{EI^2} \left[\int_0^A y^2 dA \right] dx \qquad \int_0^A y^2 dA = I$$

$$= \int_0^L \frac{Mm}{EI^2} \times I \, dx$$

$$\Delta = \int_0^L \frac{Mm \, dx}{EI}$$

Law of Reciprocal deflection or Maxwell's Reciprocal Theorem

In any beam or truss the deflection at any point D due to the load at any other point C is the same as the deflection at C due to the same load W applied at D

Consider a structure AB carrying a load W applied at any point C. Let the deflection at C be Δ_c and the deflection at any other point D be Δ_d

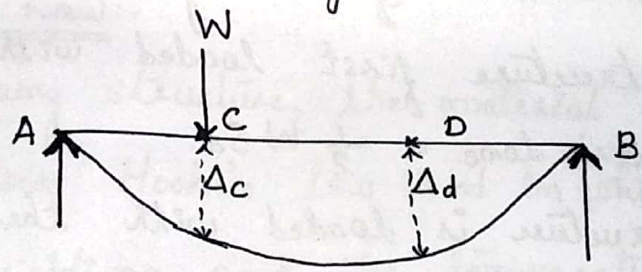


Fig: 1

Consider the same structure AB carrying the same load W at D. Let the deflection at C and D be δ_c and δ_d respectively.

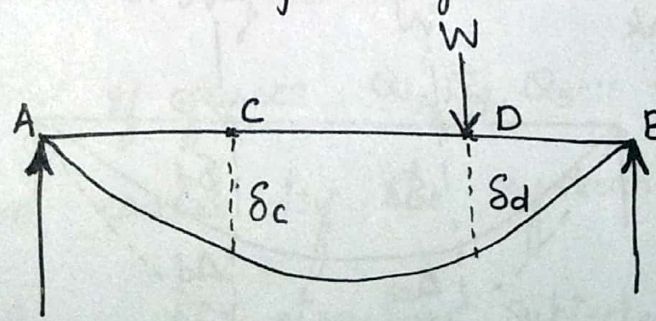
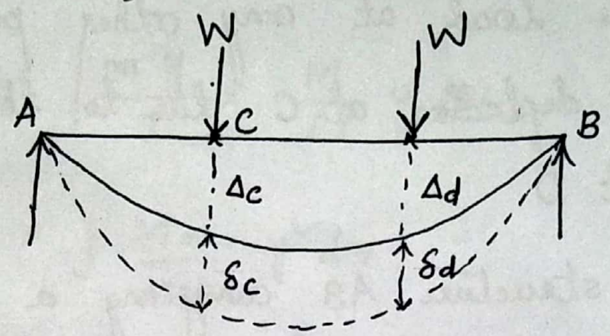


Fig. 2.

Let the structure be loaded as shown in fig. 1, ie W @ C

$\therefore$ Work done on the structure $= \frac{1}{2} W \Delta_c$

As the structure is loaded with the load W at C let another load W be applied at D. There will be further deflections δ_c and δ_d at C and D



Total work done = $\frac{1}{2} W \Delta_c + \frac{1}{2} W \delta_d + W \delta_c$

Let now the order of loading be changed,

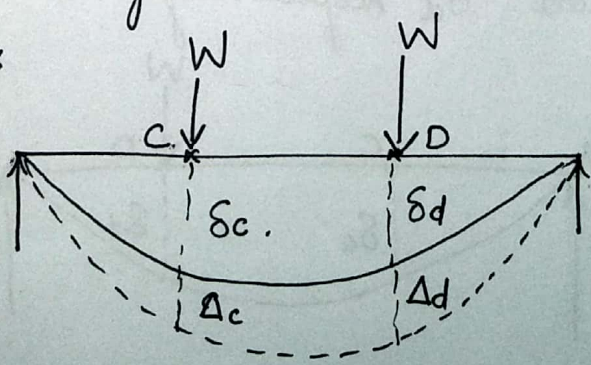
Let the structure first loaded with W at D,

then the work done = $\frac{1}{2} W \delta_d$.

As the structure is loaded with the load W at D, Let an equal load W be applied at C.

Further deflections of Δ_c and Δ_d will occur at C and D respectively.

~~Total work~~



Total work done = $\frac{1}{2} W \delta_d + \frac{1}{2} W \Delta_c + W \Delta_d$

Equating the two loads expressions obtained for the total workdone when both the loads are present on the structure

$$\therefore \frac{1}{2} W \Delta_c + \frac{1}{2} W \delta_d + W \delta_c = \frac{1}{2} W \delta_d + \frac{1}{2} W \Delta_c + W \Delta_d$$

$$\delta_c = \Delta_d.$$

$\therefore$ The deflection at C due to the load W at D = The deflection at D due to the same load W at C

Bette's Law:-

In any structure, the material of which is elastic and follows Hooke's law and in which the supports are unyielding and the temperature is constant, the virtual work done by a system of forces $P_1, P_2, P_3 \cdots$ during the distortion caused by a system of forces $Q_1, Q_2, Q_3 \cdots$ is equal to the virtual work done by the system of forces $Q_1, Q_2, Q_3 \cdots$ during the distortion caused by the system of forces. $P_1, P_2, P_3 \cdots$

Figure shows the structure subjected to seperately to the two system of forces

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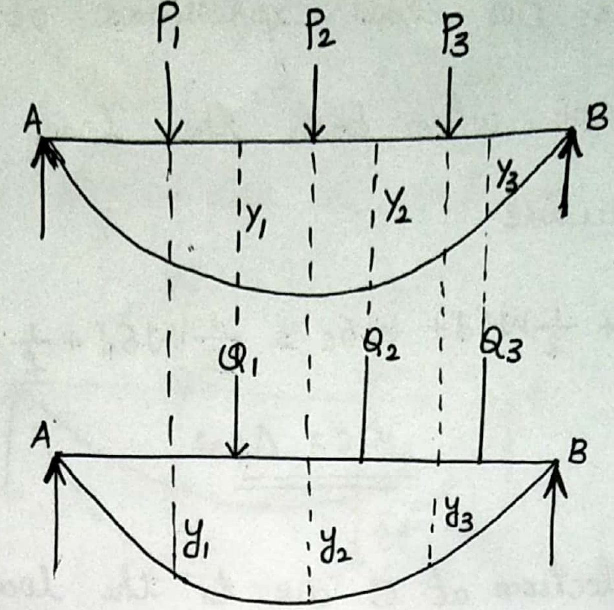


Let W_e = External workdone on the structure when the system of forces P_1, P_2, P_3 is applied

W_e' = External workdone on the structure when the system of forces Q_1, Q_2, Q_3 is applied

Let y_1, y_2, y_3 be the deflections caused by the system of forces Q_1, Q_2, Q_3 at the points of application of the forces P_1, P_2, P_3 respectively

Let Y_1, Y_2, Y_3 be the deflections caused by the system of forces P_1, P_2, P_3 at the points of application of the forces Q_1, Q_2, Q_3 respectively

Let the structure be first loaded with the system of forces P_1, P_2, P_3

$\therefore$ Work done on the structure = W_e

As the structure is carrying this system of forces, let now the system of forces Q_1, Q_2, Q_3 be applied.

Total work done = $W_e + W_e' + P_1 y_1 + P_2 y_2 + P_3 y_3$

Let now the order of loading be changed.

Let the structure be first loaded with the system of forces Q_1, Q_2, Q_3

Work done on the structure = W_e'

As the structure is carrying this system, let the system of forces P_1, P_2, P_3 be applied

Total workdone = $W_e' + W_e + Q_1 Y_1 + Q_2 Y_3 + Q_3 Y_3$

Equating the expressions for the total workdone when both the system of forces are present on the structure

We have

$$W_e + W_e' + P_1 y_1 + P_2 y_2 + P_3 y_3 = W_e' + W_e + Q_1 Y_1 + Q_2 Y_2 + Q_3 Y_3$$

$$\therefore P_1 y_1 + P_2 y_2 + P_3 y_3 = Q_1 Y_1 + Q_2 Y_2 + Q_3 Y_3$$

Virtual work done by the system of forces P_1, P_2, P_3 due to the deflection caused by the system of forces Q_1, Q_2, Q_3 equals virtual work done by the system of forces Q_1, Q_2, Q_3 due to the deflections caused by the system of forces P_1, P_2, P_3.

Statically determinate Structures	Statically Indeterminate Stru
i) Conditions of equilibrium are sufficient to fully analyse the structure	i) Conditions of equilibrium are insufficient to analyse the structure fully
ii) The BM at ~~any~~ a section or the force in any member is independent of the material of the components of the structure	ii) The BM at a section or the force in a member depends upon the material of the components of the structure
iii) The BM at a section or the force in any member is independent of the cross-sectional areas of the components	iii) The BM at a section or the force in a member depends upon the cross section areas of the components
iv) No stresses are caused due to temperature changes.	iv) Stresses are caused due to temperature changes
v) No stresses are caused due to lack of fit.	v) Stresses are caused due to lack of fit

The Second Theorem of Castigliano (Principle of least work)

In any and every case of statical indeterminateo wherein, an indefinite number of different values of the redundant forces satisfy the conditions of statical equilibrium, their actual values are those that render the total strain energy stored to a minimum

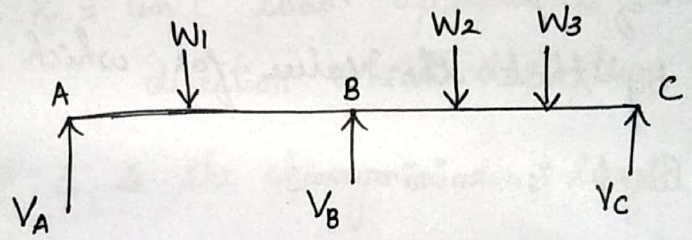


Figure shows a continous beam. If the reaction at A is X, then the conditions of equillibrium are not sufficient to find the value of X. For instance assuming any value of X, the reactions V_B & V_C can be determined. Hence ~~with~~ infinite values of X with corresponding values of V_B & V_C along with the given loads; satisfy the conditions of equilibrium.

The actual value of X can be determined from the condition that the deflection at A is zero. But by the first theorem of castigliano, the deflection at A is given by $\frac{dWi}{dx}$, where

Wi = strain energy stored by the structure

$$\therefore \frac{dWi}{dx} = 0$$

This is the condition for minimum value of Wi

Hence for all the infinite values of 'x' satisfying the conditions of equilibrium, the actual value of 'x' is given by that the value for which the strain energy stored is minimum

Deflections due to Lack of fit and Temperature changes:-

From the principle of virtual work, it has been proved that deflection of a joint can be obtained using the equation.

$$\delta = \sum \frac{PkL}{AE} = \sum k\Delta \qquad \text{where } \Delta = \frac{PL}{AE}$$

where k = unit load applied at the joint in the direction which deflection is required

Δ ie the change in the length of members need not be only due to applied loads, but may be due to any other ~~case~~. cause.

Change in length may be due to selecting imprope lengths while fabricating or may be due to temperature changes. Thus.

$$\Delta = \delta + \delta_f + \delta_t .$$

where δ = change in length due to given loadings.

δ_f = Lack of fit

δ_t = changes due to temperature

$$\delta_t = \alpha t L$$

where

α = Coefficient of thermal expansion.

t = Temperature

L = Length of the member.

Static and Kinematic Indeterminacy

Static Indeterminacy

The word redundant means extra, not required etc.

Thus when we provide more supports than the minimum required for external stability, we make the structure externally redundant.

$\Sigma F_x = 0, \ \Sigma F_y = 0, \ \Sigma M = 0$

These eqns are called eqns of static equilibrium of a planar structure subjected to general system of forces. The simultaneous solutions of these three equations will determine the magnitude of unknown reaction components. However, there are more than 3 unknown independent reaction components or elements, these 3 eqns of static equillibrium are not sufficient to determine all unknown reactions. The unknown reaction elements cannot determined simply by the equations of static equillibrium, the structure is said to be statically indeterminate, additional eqns based on compatability or deformation must be writte inorder to obtain sufficient no. of eqns for the determination of all unknown elements. The no. of these addional eqns necessary for the solution of problem is known as the degree of redundancy of the structure. The total degree of static indeterminacy of the structure denote 'Ds' may be considered as the sum of the degree of internal indeterminacy Dsi and degree of external indeterminacy Dse.

$$D_s = D_{si} + D_{se}$$

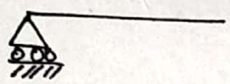
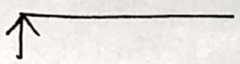
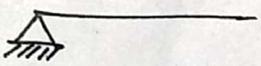
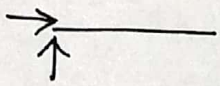
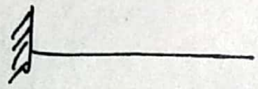
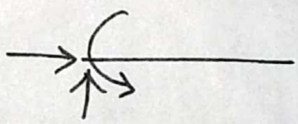
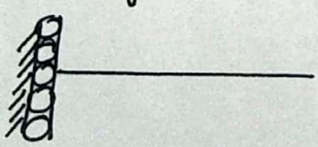
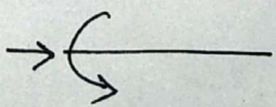
External indeterminacy is related to the support system of the structure and internal indeterminacy is related to the number of members on the structure

Kinematic Indeterminacy

A skeltal structure is said to be a kinematically indeterminate, if the displacement components of its joints cannot be determined by compatability eqns alone. Inorder to evaluate the displacement components at the joints of these structures, it is necessary to consider the equillibrium eqns. In the case of a kinematically indeterminate structure, the number of unknown displacement components are greater than the no. of compatability eqns.

The number of additional equilibrium eqns necessary for the determination of all the independent displacement components are also known as kinemati Indeterminacy or the degree of freedom of the structure

Type of supports

Type of support	Reactions.	Displacements/DOF
1) Roller 	One Reaction. 	Two < 1. Translation 2. Rotation
2) Pinned 	Two Reactions 	One DOF 1. Rotation
3) Fixed 	Three Reactions. 	ZERO DOF
4) Sliding Support 	Two Reactions. 	One DOF 1. Translation

Pin jointed Frames :-

SID = $m + r - 2n$. (plane frame or 2D frame)

KID = $2j - e$ "

SID = $m + r - 3n$ (space frame or 3D frame)

KID = $3j - e$ "

where

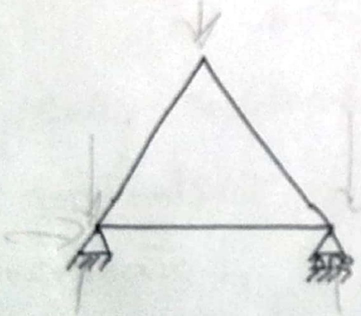
m = no of members.

r = no. of Reactions.

n = no. of joints

j = no. of joints.

e = no of compatability conditions known



SID = $m + r - 2n$ = $3 + 3 - 2 \times 3 = 0$

KID = $2j - e$ → $2 \times 3 - 3 = 3$

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Rigid jointed Frames:-

Plane or 2D frame

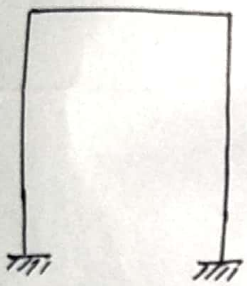
$$SID = (3m+r) - 3n$$

$$KID = 3j - e$$

Space or 3D frame

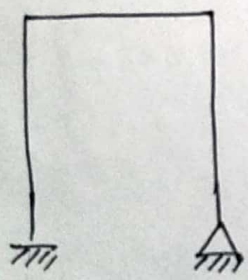
$$SID = (6m+r) - 6n$$

$$KID = 6j - e$$



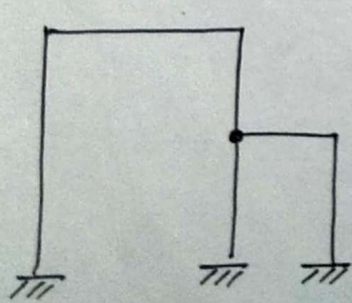
$$SID = (3m+r) - 3n = (3\times3+6) - 3\times4 = 3$$

$$KID = 3j - e = 3\times4 - 6 = 6$$



$$SID = (3m+r) - 3n = (3\times3+5) - 3\times4 = 2$$

$$KID = 3j - e = 3\times4 - 5 = 7$$



$$SID = [(3m+r) - 3n] - 1$$

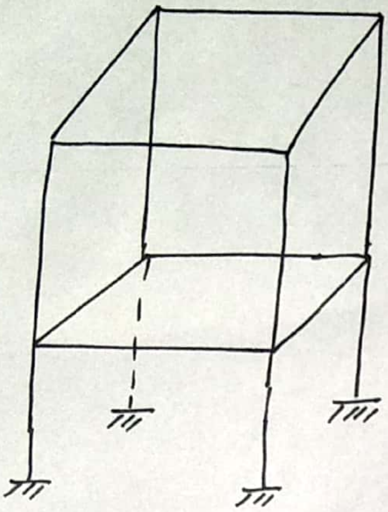
$$= [3\times6 + 9 - 3\times7] - 1 = 5$$

$$KID = [3j - e] + 1 \qquad e = m + r$$

$$= 6 + 9 = 15.$$

$$= [(3\times7) - 15] + 1$$

$$= 7$$



SID = $6m + r - 6n$

$= (6 \times 16 + 12) - 6 \times 12 = 36$

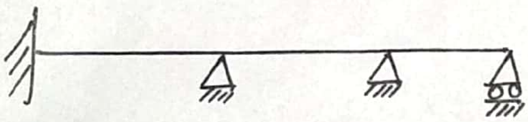
KID = $6j - e$

$= 6 \times 12 - 24 = 48$

Beams:

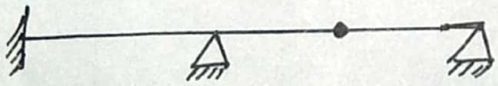
SID = $3m + r - 3n$

KID = $3j - e$.



SID = $3 \times 3 + 8 - 3 \times 4 = 5$

KID = $3j - e = 3 \times 4 - 8 = 4$



SID = $[3m + r - 3n] - 1$

$= [3 \times 3 + 7 - 3 \times 4] - 1 = 3$

KID = $[3j - e] + 1$

$= [3 \times 4 - (7+3)] + 1 = 3$

for hinge $e = m + r$

For the analysis of indeterminate structures

i- Force method

ii- Displacement Method

Force Method	Displacement method
1. Unknowns – forces	Displacements are the unknow
- Additional compatability eqns are reqd.	- Additional equilibrium eqn are required
∴ Compatability method	Equilibrium method
- Flexibility method	- Stiffness method